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Classification of bicovariant differential calculi

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Abstract

We show that the bicovariant first-order differential calculi on a factorisable quantum group with the Peter–Weyl decomposition property are in 1–1 correspondence with irreducible representations V of the quantum group enveloping algebra. The corresponding calculus is constructed and has dimension $\dim V^2$. The differential calculi on a finite group algebra $\mathbb{C}G$ are also classified and shown to be in correspondence with pairs consisting of an irreducible representation V and a continuous parameter in $\mathbb{C}P^{\dim V-1}$. They have dimension $\dim V$. For a classical Lie group we obtain an infinite family of non-standard calculi. General constructions for bicovariant calculi and their quantum tangent spaces are also obtained. © 1998 Elsevier Science B.V.

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1. Introduction

One of the first steps in non-commutative geometry of the kind coming out of quantum groups is the choice of 'first-order differential calculus' or 'cotangent bundle'. Only when this is fixed can one begin to do gauge theory [1] or make other geometrical constructions. When the quantum space in question is a quantum group, A, it is natural to require that

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the differential calculus is covariant under left and right translations. Thus, we require $\Gamma \equiv \Omega^1(A)$, $\mathrm{d}: A \to \Gamma$ such that

(1) Γ is an A-bimodule.

(2) Γ is an A-cobimodule, with coactions $\Delta_{\mathrm{L}}: \Gamma \to A \otimes \Gamma$, $\Delta_{\mathrm{R}}: \Gamma \to \Gamma \otimes A$ bimodule maps.

(3) $\mathrm{d}: A \to \Gamma$ is a bicomodule map.

(4) $\mathrm{d}(ab) = (\mathrm{d}a)b + a(\mathrm{d}b)$ for all $a, b \in A$.

(5) $\Gamma = \mathrm{span}\{a\,\mathrm{d}b \mid a, b \in A\}$.

Here, a bicomodule is like a bimodule but with arrows reversed, i.e. a pair of commuting coactions $\Delta_{\mathrm{L}}, \Delta_{\mathrm{R}}$. A morphism of differential calculi means a bimodule and bicomodule map forming a commutative triangle with the d maps. These are the natural axioms studied by Woronowicz [2]. The axiom (5) here forces the calculus to be irreducible, and is assumed throughout. It should not be confused with a further coirreducibility condition which we will impose later. By now, several examples of bicovariant calculi are known, as well as a limited classification of further variants of the known calculi of a particular form [3]. Among general constructions, the class of 'inner' bicovariant calculi has also been introduced [4]. The classification of all the possible calculi on a general quantum group, however, has remained open until now.

The main result of the paper, in Section 4, is a complete solution to this classification problem under the assumption of a factorisable quantum group with the Peter–Weyl decomposition property. The standard q-deformed function algebras G_q of semisimple Lie groups are essentially of this type, up to suitable localisations or when working over formal power-series. In this case our algebraic result constructs a calculus on G_q of dimension $(\dim V)^2$ for each irreducible representation V of $U_q(g)$, and indicates that these are the only 'generic' possibilities in the sense of extending to localisations or to working over formal power-series in the deformation parameter.

We begin in Section 2 with some general constructions for bicovariant calculi on arbitrary Hopf algebras. We use, in fact, a dual reformulation of [2] in terms of 'quantum tangent spaces' (rather than 1-forms) as subrepresentations of a particular quantum double representation. After recalling the preliminary material, we obtain some new results from this point of view. For example, we construct quantum tangent spaces associated to arbitrary central elements in the quantum group enveloping algebra. This provides for all G_q a natural deformation of the classical calculus, by using the quadratic q-Casimir of the corresponding $U_q(g)$.

In Section 3 we apply the general results to the complete classification for $A = \mathbb{C}G$, the group algebra over a finite group, as well as recovering the known classification for the function algebra $A = \mathbb{C}(G)$. We also comment on the case where G is a Lie group and show that the Casimir construction in this case (with the quadratic Casimir of $U(g)$) recovers the standard commutative differential calculus. Section 5 concludes with some directions for further work.

We work over $\mathbb{C}$ for convenience, but all abstract Hopf-algebraic results work over any ground field or, with care, over a ring such as $\mathbb{C}[[\hbar]]$.

2. Quantum tangent spaces and some constructions for them

We begin with some preliminary material recalling the well-known role of the quantum double in classifying bicovariant differential calculi [2], albeit reformulated in dual terms, as quantum tangent spaces. It is these which will actually be constructed and classified in the paper. The preliminary material (through to Proposition 2.4) should essentially be known to experts, although we have not found a suitable treatment elsewhere. Proposition 2.3 is perhaps more novel and completes the 'quantum tangent space' picture (as braided derivations). We then proceed to the main new results of the section, such as the Casimir construction and a 'mirror' operation on the moduli space of bicovariant differential calculi.

Let H be a Hopf algebra non-degenerately dually paired with A. The Drinfeld quantum double [5] is the double cross product Hopf algebra $H\bowtie A^{\rm op}$ built on $H\otimes A$ with the product

$$(h\otimes a)(g\otimes b) = hg_{(2)}\otimes ba_{(2)}\langle g_{(1)}, a_{(1)}\rangle\langle g_{(3)}, Sa_{(3)}\rangle, \quad a, b\in A,\ h, g\in H \tag{1}$$

and tensor product unit and coalgebra. This is the formulation from [6]. We use here (and throughout) the notations and conventions from [7]. Thus, $\Delta h = h_{(1)}\otimes h_{(2)}$ is the coproduct, S is the antipode (which we assume for convenience to be invertible), and $\langle\ ,\ \rangle$ is the pairing between H and A. We denote the counit of any of our Hopf algebras by ϵ.

The quantum double has a formal quasitriangular structure $\mathcal{R} = \sum_a f^a\otimes e_a$ where $\{e_a\}$ is a basis of H and $\{f^a\}$ a dual basis of A. Although formal, this does lead to a braiding Ψ among suitable representations. To make this precise, we define a representation of the quantum double of H to be A-regular if the action of $A\subset H\bowtie A^{\rm op}$ is given by evaluation against a (left) coaction of H. It is H-regular if the action of H is given by evaluation against a (right) coaction of A. If V is A-regular or W is H-regular then $\Psi : V\otimes W\to W\otimes V$ is a well-defined operator. Thus, $\Psi(v\otimes w) = \sum_a e_a{\triangleright} w\otimes f^a{\triangleright} v = \sum_a e_a{\triangleright} w\otimes\langle f^a, v^{\bar{(1)}}\rangle v^{\bar{(2)}} = v^{\bar{(1)}}{\triangleright} w\otimes v^{\bar{(2)}}$ in the first case, where $v\mapsto v^{\bar{(1)}}\otimes v^{\bar{(2)}}$ (with summation understood) is the assumed coaction $V\to H\otimes V$. Similarly in the second case.

Woronowicz [2] observed that first-order bicovariant calculi are in 1–1 correspondence with Ad-invariant 'ideals' in $\ker\epsilon$. More precisely, they correspond to quotients of $\ker\epsilon\subset A$ by subspaces M which are stable under the action and coaction

$$a{\triangleright} v = av,\quad \mathrm{Ad}(v) = v_{(1)}Sv_{(3)}\otimes v_{(2)} \tag{2}$$

on $v\in\ker\epsilon$. Equivalently, they correspond to quotients V to which this action and coaction descend. The corresponding calculus is $\Gamma = V\otimes A$ with tensor product action and coaction from the left and trivial action and coaction on V from the right (here we take the left and right actions and coactions on A defined by its product and coproduct). In addition, ${\rm d}a = a_{(1)}\otimes a_{(2)} - 1\otimes a$, with the first tensor factor here projected to V. This is the most general form for a bicovariant calculus up to isomorphism. The case $V = \ker\epsilon\subset A$ is called the 'universal' first-order calculus.

As a first step, we can write all coactions of A as actions of H. Then (2) becomes

$$a{\triangleright} v = av,\quad h{\triangleright} v = \langle h, v_{(1)}Sv_{(3)}\rangle v_{(2)}\quad \forall a\in A,\ h\in H, \tag{3}$$

and calculi correspond to quotients of which are equivariant under these actions. The quantum double is not needed to classify calculi here, but in fact these two actions do fit together to form a representation of $A{\bowtie}H^{\rm op}$, the quantum double of A. This fact allows one to deduce, for example, the canonical braiding σ in [2] from the quantum double quasitriangular structure, which would otherwise have to be introduced by hand. This was explained in [8]. In this context, it is natural also to reformulate the bicomodules $\Delta_{\rm L}$, $\Delta_{\rm R}$ in the axioms (2) and (3) of a bicovariant calculus as an $H^{\rm op}$-bimodule by evaluation against the coactions. In principle, requiring an $H^{\rm op}$-bimodule could be slightly more general when H is infinite-dimensional.

Lemma 2.1. *The quantum double* $H{\bowtie}A^{\rm op}$ *acts on* $\ker\epsilon\subset H$ *by*

$$h{\triangleright}x=h_{(1)}xSh_{(2)},\qquad a{\triangleright}x=\langle a,x_{(1)}\rangle x_{(2)}-\langle a,x\rangle 1$$

for all $x\in\ker\epsilon\subset H$ *and* $a\in A$, $h\in H$.

Proof. The quantum double has a well-known 'Schroedinger' representation on H [7] by the quantum adjoint action and by the coregular 'differentiation' representation. This induces the action stated on $\ker\epsilon$ via the projection $\Pi(h)=h-1\epsilon(h)$ as a morphism $H\to\ker\epsilon$, i.e. it is easy to see that it is indeed an action of the quantum double on $\ker\epsilon$ and that Π is an intertwiner. Also, we can identify the linear space $\ker\epsilon\subset A$ with $A/\mathbb{C}$ (the quotient by the one-dimensional vector space spanned by the unit element); for any element in $A/\mathbb{C}$ there is a unique representative in $\ker\epsilon\subset A$. In terms of $A/\mathbb{C}$ the action in (3) is

$$a{\triangleright}v=av-\epsilon(v)a\qquad h{\triangleright}v=\langle h,v_{(1)}Sv_{(3)}\rangle v_{(2)},\qquad \forall v\in A/\mathbb{C},\ a\in A,\ h\in H. \tag{4}$$

The action stated in the lemma is the natural right action of $A{\bowtie}H^{\rm op}$ on $\ker\epsilon\subset H$ dual to this action on $A/\mathbb{C}$, viewed as a left action of the quantum double of H. □

We are now ready to make a further reformulation which, when the bicovariant calculus is finite-dimensional as an A-module, is strictly equivalent by dualising V.

Proposition 2.2. *Finite-dimensional bicovariant calculi are in* 1–1 *correspondence with subrepresentations* $L\subseteq\ker\epsilon\subset H$ *of the quantum double representation in Lemma* 2.1*:*

$$\Gamma={\rm Lin}(L,A),\qquad ({\rm d}a)(x)=\langle x,a_{(1)}\rangle a_{(2)},$$
$$(a\cdot\gamma)(x)=a_{(2)}\gamma(a_{(1)}{\triangleright}x),\qquad (\gamma\cdot a)(x)=\gamma(x)a,$$
$$(h\cdot\gamma)(x)=\langle h_{(2)},\gamma(h_{(1)}{\triangleright}x)_{(1)}\rangle\gamma(h_{(1)}{\triangleright}x)_{(2)},\qquad (\gamma\cdot h)(x)=\gamma(x)_{(1)}\langle\gamma(x)_{(2)},h\rangle$$

for all $\gamma\in{\rm Lin}(L,A)$, $a\in H$ *and* $h\in H$. *The vector space* L *is called the* quantum tangent space *associated to the calculus.*

Proof. A quotient of $A/\mathbb{C}$ which is equivariant under the quantum double action corresponds under dualisation to a subspace of $\ker\epsilon\subset H$ which is stable under the action in Lemma 2.1.

Thus, this is a dual formulation of the correspondence (3). Also, with the projection to V understood, we have $\mathrm{d}a = a_{(1)}\otimes a_{(2)} - 1\otimes a = a_{(1)}\otimes a_{(2)}$ when we work with V as a quotient of $A/\mathbb{C}$. This leads to the form shown for d. It is easy to verify directly that the structures shown indeed provide a first-order bicovariant differential calculus given L. Conversely, in the finite-dimensional case, we define $L = V^*$, where V is the invariant part of Γ under the usual correspondence in (3). In terms of the ideal M which defines the bicovariant calculus via (2), the corresponding quantum double subrepresentation is $L = \{x \in \ker\epsilon \mid \langle x, a\rangle = 0\ \forall a \in M\}$. □

The correspondence in Proposition 2.2 is contragradient, i.e. morphisms of calculi $\Gamma_1 \to \Gamma_2$ correspond to inclusions $L_2 \hookrightarrow L_1$ of quantum double subrepresentations. Only inclusions are allowed here, corresponding to all morphisms of calculi being of the form Γ_2 a quotient of Γ_1. In the infinite-dimensional case every bicovariant calculus continues to define a subrepresentation L, and, conversely, a subrepresentation L continues to provide a bicovariant first-order calculus in our slightly generalised sense where an action of H^{op} replaces the coactions $\Delta_{\mathrm{L}}, \Delta_{\mathrm{R}}$. It is this final reformulation in terms quantum tangent spaces which we will use; by definition a quantum tangent space L is a subrepresentation of $\ker\epsilon \subset H$ under the action of the quantum double of H, and it is these which we will actually classify in the present paper. Indeed, quantum tangent spaces are an equally good starting point for differential calculus.

Proposition 2.3. *For each $x \in L$, we define the 'braided derivation'*

$$\partial_x : A \to A, \qquad \partial_x(a) = (\mathrm{d}a)(x).$$

This obeys

$$\partial_x(ab) = (\partial_x a)b + \Psi(a\otimes x)^{-(2)}\partial_{\Psi(a\otimes x)^{-(1)}}b,$$

where $\Psi : L\otimes A \to A\otimes L$ is the quantum double braiding between A, L as quantum double modules, with inverse $\Psi^{-1}(a\otimes x)$ denoted explicitly by $\Psi(a\otimes x)^{-(1)}\otimes\Psi(a\otimes x)^{-(2)}$.

Proof. We start with the identity

$$\begin{aligned}\partial_x(ab) &= \langle x, (ab)_{(1)}\rangle(ab)_{(2)} = \langle x_{(1)}, a_{(1)}\rangle\langle x_{(2)}, b_{(1)}\rangle a_{(2)}b_{(2)}\\ &= \langle x, a_{(1)}\rangle a_{(2)}b + \langle a_{(1)}\triangleright x, b_{(1)}\rangle a_{(2)}b_{(2)} = \partial_x(a)b + a_{(2)}\partial_{a_{(1)}\triangleright x}(b)\end{aligned}$$

based on the definition of ∂_x and the action in Lemma 2.1. On the other hand, A is a quantum double module by

$$h\triangleright a = \langle Sh, a_{(1)}\rangle a_{(2)}, \quad b\triangleright a = (S^{-1}b_{(2)})ab_{(1)}, \tag{5}$$

which is the conjugate (dual) of the Schroedinger representation of the quantum double on H. It is H-regular, so that the braiding Ψ is well-defined. We compute it easily as

$$\Psi(x\otimes a) = e_a\triangleright a\otimes f^a\triangleright x = a_{(2)}\otimes Sa_{(1)}\triangleright x \tag{6}$$

with inverse $\Psi^{-1}(a\otimes x) = a_{(1)}\triangleright x\otimes a_{(2)}$, which we put into the above identity. □

Because the subspace L is stable under the quantum adjoint action, it is tempting to restrict the latter to L as a 'quantum Lie bracket' $[\ ,\ ] = \mathrm{Ad} : L\otimes L \to L$. The use of Ad as 'quantum Lie bracket' has been discussed in [2] from this point of view and independently from another point of view in [9] (where the 'quantum Lie bracket structure constants' for the l^+Sl^- generators of $U_q(g)$ were computed in R-matrix form). The only content here comes from the identities

$$[x, [y, z]] = [[x_{(1)}, y], [x_{(2)}, z]], \quad [x_{(1)}, y]x_{(2)} = xy \tag{7}$$

for any quantum group with $[\ ,\ ] = \mathrm{Ad}$ the left adjoint action. For completeness, cf. [2]:

Proposition 2.4. *The 'quantum Lie bracket' on L defined by* Ad *obeys*

$$[x, [y, z]] = [[x, y], z] + [\ , [\ , z]] \circ \Psi(x\otimes y), \quad [x, y] = xy - \cdot\Psi(x\otimes y),$$

where Ψ is the quantum double braiding between L, L as quantum double modules.

Proof. We again use the formula $\Psi(x\otimes y) = e_a\triangleright y\otimes f^a\triangleright x$; our action of the quantum double of H is regular and we still have a well-defined operator

$$\Psi(x\otimes y) = \mathrm{Ad}_{e_a}(y)\otimes(\langle f^a, x_{(1)}\rangle x_{(2)} - \langle f^a, x\rangle 1) = [x_{(1)}, y]\otimes x_{(2)} - [x, y]\otimes 1. \tag{8}$$

Then (7) can be trivially rewritten in the form stated by eliminating the coproduct in favour of Ψ in these equations. □

We would like to stress that most L, $[\ ,\ ]$ are not, however, reasonable as a 'quantum Lie algebra' of H; one needs further structure on L for this. The problems have been explained in [9] and lead one to the braided version [10]:

(1) Although 'enveloping algebra-like' relations $[x, y] = xy - \cdot\Psi(x\otimes y)$ hold in H, we do not know that L generates H. Even if it does, we do not know that these are the only relations in H. Indeed, for $U_q(g)$ they are not. So $H \neq U(L)$ as generated by L and such relations.

(2) Even if H were to be generated in some way from L, we are not able to recover the coproduct of H in this way. Indeed, for $U_q(g)$ the coproduct of H does not have any simple form on L and hence cannot be generated in some canonical way. Without this, $U(L)$ is only an algebra and not a Hopf algebra or bialgebra. Equivalently, one cannot tensor product representations of L in any natural way, which makes it useless as a 'Lie algebra' symmetry.

In the case where H is quasitriangular, there is a 'transmutation theory' [11] which converts H to a braided group. It also converts L to a 'braided-Lie algebra' $\mathcal{L}$. The linear maps $[\ ,\ ]$ are the same (the braided adjoint action coincides with the quantum one) but the coalgebras are different. For the standard calculus on G_q, the braided coproduct takes a standard matrix form on $\mathcal{L}$ and there is a corresponding $U(\mathcal{L})$ as a braided group (bialgebra in a braided category) generated from $\mathcal{L}$. Thus, the problems (1) and (2) are resolved at the price of working with the braided version of the theory. But for a general Hopf algebra H and general L we do not really have a 'quantum Lie algebra' or braided-Lie algebra at all. Hence we prefer the term 'quantum tangent space' for the subspace L.

We would also like to stress that not all quantum double modules are relevant to the constructions above, but only the modules V which are quotients of $\ker\epsilon \subset A$, or their duals $L = V^*$ which (as explained above) are subrepresentations of $\ker\epsilon \subset H$. Moreover, in the infinite-dimensional case one has various formulations which are not quite equivalent. If one wants to work entirely with the Hopf algebra A and differential forms, one may work throughout with crossed A-modules (i.e. an action and coaction of A). It is known from [6] that these coincide with $D(A)$-modules in the finite-dimensional case. If one is interested only in quantum tangent spaces one can work entirely with H, by working with $D(H)$-modules as crossed H-modules (i.e. an action and coaction of H). The above formulation with A and H dually paired is a third option. Of these formulations, the one in terms of crossed A-modules follows most directly from the axioms: following [12], it is clear that a bicovariant bimodule Γ as in axioms (1) and (2) of a bicovariant calculus (also known as bi-Hopf module) is the same thing as a crossed module $V = \Gamma^{\rm inv}$ (the A-coinvariant part of Γ), which is the point of view used by Woronowicz [2].

We proceed now to general constructions for bicovariant calculi and their quantum tangent spaces. Firstly, we recall that any element $\alpha \in A$ which is invariant under the adjoint coaction Ad in (2) can be used to generate an ideal $A(\alpha - \epsilon(\alpha))$ to quotient $\ker\epsilon \subset A$ by. This class of bicovariant calculi can be called *inner type-I* because the exterior derivative obeys

$$\begin{aligned}\epsilon(\alpha){\rm d}a &= a_{(1)}\epsilon(\alpha)\otimes a_{(2)} - 1\epsilon(\alpha)\otimes a\\ &= a_{(1)}\alpha\otimes a_{(2)} - \alpha\otimes a = a\cdot(\alpha\otimes 1) - (\alpha\otimes 1)\cdot a \qquad (9)\end{aligned}$$

projected down to the quotient. The expression on the right is shown lifted up to $A\otimes A$ as a left A-module by the tensor product left action and a right A-module by right multiplication in the second copy. A variant of this construction was introduced in [4], where we quotient by the ideal $(\ker\epsilon).(\alpha - (\epsilon(\alpha) + 1)) \subseteq \ker\epsilon \subset A$ and we have

$$\begin{aligned}{\rm d}a &= (a_{(1)} - \epsilon(a_{(1)}))\otimes a_{(2)}\\ &= a_{(1)}(\alpha - \epsilon(\alpha))\otimes a_{(2)} - (\alpha - \epsilon(\alpha))\otimes a = a\cdot\omega(\alpha) - \omega(\alpha)\cdot a \qquad (10)\end{aligned}$$

in the quotient, where $\omega(\alpha) = (\alpha - \epsilon(\alpha))\otimes 1 = ({\rm d}\alpha_{(1)})S\alpha_{(2)} \in \Gamma$. This class can be called *inner type-II*. Since Ad-invariant elements α are closed under addition and multiplication, we have whole ring of bicovariant differential calculi of either type. The standard calculi on G_q were already obtained in [13] as a quotient of an inner type-II form (with α the q-trace), while Brzeziński and Majid [4] extended this to a ring of calculi generated by $\alpha_1, \ldots, \alpha_r \in G_q$ constructed by transmutation.

Note that the condition for a general calculus that V is an equivariant quotient of $\ker\epsilon \subset A$ is the same thing as an equivariant surjection $\pi : A/\mathbb{C} \to V$, where the quantum double acts as in (4). This is the same thing as a surjection $\pi : A \to V$ obeying

$$\pi(ab) = \epsilon(b)\pi(a) + a{\triangleright}\pi(b) \quad \forall a, b \in A, \qquad (11)$$

and intertwining the action of H (or coaction of A), i.e. a surjective equivariant Hocschild 1-cocycle, cf. [14] from a different point of view. This provides a class of *coboundary calculi* where π is a Hocschild coboundary, i.e. $\pi(a) = a \triangleright \nu - \epsilon(a)\nu$ for some invariant $\nu \in V$. Surjectivity means that we are forced to $\nu = \pi(\alpha)$ for some $\alpha \in A$ which is Ad-invariant up to projection by π. One has $\omega(\alpha) = (\mathrm{d}\alpha_{(1)})S\alpha_{(2)} = \pi(\alpha) \otimes 1$ and

$$\mathrm{d}a = \pi(a_{(1)}) \otimes a_{(2)} = a_{(1)} \triangleright \pi(\alpha) \otimes a_{(2)} - \pi(\alpha) \otimes a = a \cdot \omega(\alpha) - \omega(\alpha) \cdot a \quad \forall a \in A \tag{12}$$

since $\pi(a) = a \triangleright \pi(\alpha) - \epsilon(a)\pi(\alpha)$. The inner type-II construction from [4] is an example of this more general coboundary type: writing the projection map π for this explicitly, $\pi(a)(1 + \epsilon(\alpha)) = \pi((a - \epsilon(a))(1 + \epsilon(\alpha)) = \pi((a - \epsilon(a))\alpha) = \epsilon(\alpha)\pi(a) + a \triangleright \pi(\alpha) - \epsilon(a)\pi(\alpha)$ since π sends $\ker \epsilon \cdot (\alpha - (1 + \epsilon(\alpha)))$ to zero and obeys (11). This is a little different from a corresponding discussion in [14].

Proposition 2.5. *The quantum tangent space for an inner type-I bicovariant differential calculus defined by any non-trivial element $\alpha \in A$ invariant under* Ad *in* (2) *is the quantum double subrepresentation*

$$L_\alpha = \{x \in \ker \epsilon \subset H \mid x_{(1)} \langle x_{(2)}, \alpha \rangle = x\epsilon(\alpha)\}.$$

This has a canonical extension $\tilde{L}_\alpha$ where the condition on x is only required to hold on evaluation against all $a \in \ker \epsilon \subset A$. Similarly, the quantum tangent space for the inner type-II case is

$$L_{\alpha,1} = \{x \in \ker \epsilon \subset H \mid \langle x, a\alpha \rangle = \langle x, a \rangle (\epsilon(\alpha) + 1) \forall a \in \ker \epsilon \subset A\}.$$

Proof. It is convenient to first identify $\ker \epsilon$ with $A/\mathbb{C}$ as in the proof of Lemma 2.1. Then an inner type-I bicovariant calculus has V the quotient by the image $A \triangleright \alpha$ in $A/\mathbb{C}$. Hence its dual consists of the linear functionals $x \in \ker \epsilon \subset H$ such that $\langle x, a \triangleright \alpha \rangle = 0$ for all a, i.e. such that $\langle x, a\alpha - a\epsilon(\alpha) \rangle = 0$. This leads to the dual formulation; we define L_α as stated and verify directly that it is stable under the quantum double action in Lemma 2.1, which is a straightforward Hopf algebra computation. The variant in which we require $\langle x, a\alpha \rangle = \langle x, a \rangle \epsilon(\alpha)$ for all $a \in \ker \epsilon \subset A$ is easily verified to also form a quantum double subrepresentation, and defines $\tilde{L}_\alpha$. The type-II case is similar. □

Note that we can define $L_{\alpha,\lambda}$ similarly, with $\epsilon(\alpha) + \lambda$ in place of $\epsilon(\alpha) + 1$ and then obtain $\lambda \mathrm{d}a = a \cdot \omega(\alpha) - \omega(\alpha) \cdot a$ as in [4]. All non-zero λ are equivalent to the inner type-II construction via $L_{\alpha,\lambda} = L_{\lambda^{-1}\alpha,1}$, while $L_{\alpha,0} = \tilde{L}_\alpha$. More generally, we can restrict the condition $\forall a \in \ker \epsilon \subset A$ by requiring only $a \in M$, where $M \subseteq \ker \epsilon \subset A$ is the quotienting ideal for any given bicovariant calculus. This gives a 1-parameter family of new calculi with quotienting ideal $M \cdot (\alpha - (\epsilon(\alpha) + \lambda))$, and is the general idea behind the above constructions.

The representation-theoretic point of view suggests, however, a different type of general construction for any Hopf algebra. Namely, pick any element $x \in \ker \epsilon \subset H$ and define

$L = (H{\bowtie}A^{\rm op}){\triangleright}x$, the image of x under the quantum double action. It evidently forms a subrepresentation of $\ker\epsilon$ and hence by Proposition 2.2 it defines a bicovariant differential calculus. More generally, the image of any left ideal of the quantum double acting on any $x \in \ker\epsilon \subset H$ will be a subrepresentation. An interesting special case of this idea is the following.

Proposition 2.6. *Let $c \in H$ be any non-trivial central element. There is an associated bicovariant differential calculus with*

$$L_c = \mathrm{span}\{x_a = \langle a, c_{(1)}\rangle c_{(2)} - \langle a, c\rangle 1 \mid a \in \ker\epsilon \subset A\},$$
$$\partial_{x_a}(b) = \langle ab_{(1)}, c\rangle b_{(2)} - \langle a, c\rangle b, \quad \Psi^{-1}(b\otimes x_a) = x_{ab_{(1)}}\otimes b_{(2)},$$
$$[x_a, x_b] = x_{b_{(2)}}\langle a(Sb_{(1)})b_{(3)}, c\rangle - x_b\langle a, c\rangle, \quad \Psi(x_a\otimes x_b) = x_{b_{(2)}}\otimes x_{a(Sb_{(1)})b_{(3)}}$$

for all $b \in A$ in the middle line. We say that the quantum tangent space L_c or its corresponding bicovariant differential calculus is centrally generated. *It has a canonical extension $\tilde{L}_c$ spanned by $\{x_a\}$ for all $a \in A$.*

Proof. The quantum double can also be written as $A^{\rm op}{\bowtie}H$, i.e. every element can be written uniquely in the form $\sum_i a_i h_i$ with $a_i \in A$ and $h_i \in H$. A central element c is precisely an element for which $h{\triangleright}c = \epsilon(h)c$ for all $h \in H$. Hence the image of $x = c - \epsilon(c)$ under the quantum double action reduces to the image of the action of A in Lemma 2.1. Thus $\tilde{L}_c = \mathrm{span}\{x_a = a{\triangleright}(c - \epsilon(c)) | a \in A\}$ is a subrepresentation under the quantum double. We then restrict the allowed $\{x_a\}$ to $a \in \ker\epsilon \subset A$. It is easy to check that this still defines a quantum double subrepresentation, which is the one stated. It can sometimes coincide with $\tilde{L}_c$.

The calculation of ∂_{x_a} and Ψ^{-1} from Proposition 2.3 is trivial. For the quantum Lie bracket in Proposition 2.4, we note first the Ad-invariance identity

$$c_{(1)}\otimes h_{(1)}c_{(2)}Sh_{(2)} = (Sh_{(1)})c_{(1)}h_{(2)}\otimes c_{(2)} \quad \forall h \in H, \tag{13}$$

which holds for any central element c. Then

$$\begin{aligned}\mathrm{Ad}_h(x_a) &= \langle a, c_{(1)}\rangle \mathrm{Ad}_h(c_{(2)}) - \epsilon(h)\langle a, c\rangle \\ &= \langle a, (Sh_{(1)})c_{(1)}h_{(2)}\rangle c_{(2)} - \epsilon(h)\langle a, c\rangle \\ &= x_{a_{(2)}}\langle h, (Sa_{(1)})a_{(3)}\rangle + \langle a, (Sh_{(1)})ch_{(2)}\rangle - \epsilon(h)\langle a, c\rangle \\ &= x_{a_{(2)}}\langle h, (Sa_{(1)})a_{(3)}\rangle,\end{aligned}$$

where the last two terms cancel because c is central. In other words, the map $A \to H$ sending $a \mapsto (a\otimes\mathrm{id})\Delta c$ intertwines the quantum adjoint action and the quantum coadjoint action; likewise for its projection to $\ker\epsilon$, which is the map $a \mapsto x_a$. Using this observation, we have the quantum Lie bracket

$$[x_a, x_b] = x_{b_{(2)}}\langle x_a, (Sb_{(1)})b_{(3)}\rangle = x_{b_{(2)}}\langle a, c_{(1)}\rangle\langle c_{(2)}, (Sb_{(1)})b_{(3)}\rangle - x_b\langle a, c\rangle$$

giving the result as stated. Likewise, the braiding in Proposition 2.4 comes out as

$$\begin{aligned}\Psi(x_a\otimes x_b) &= \langle a, c_{(1)}\rangle \mathrm{Ad}_{c_{(2)}}(x_b)\otimes c_{(3)} - \langle a, c_{(1)}\rangle \mathrm{Ad}_{c_{(2)}}(x_b)\otimes 1\\ &= x_{b_{(2)}}\otimes c_{(2)}\langle a(Sb_{(1)})b_{(3)}, c_{(1)}\rangle - x_{b_{(2)}}\otimes 1\langle a(Sb_{(1)})b_{(3)}, c\rangle\\ &= x_{b_{(2)}}\otimes x_{a(Sb_{(1)})b_{(3)}},\end{aligned}$$

again using the intertwining property for the map $a \mapsto x_a$. □

The centrally generated calculi are dual in a certain sense to inner type-I calculi. Thus, the quantum tangent space for the inner calculus in Proposition 2.5 can be viewed as the kernel under differentiation for a suitable (right-handed) calculus on H, taken along the direction of $\alpha - \epsilon(\alpha)$. By contrast, the quantum tangent space for a centrally generated calculus can be viewed as the projection to $\ker\epsilon \subset H$ of the image of c under differentiation along all possible $a \in \ker\epsilon \subset A$, for a suitable (left-handed) calculus on H. Also, for factorisable quantum groups (see the next section) there is a correspondence between central elements and elements invariant under a right handed Ad_R coaction, via the quantum Killing form, see [4]. So centrally generated calculi and (a right-handed version of) inner calculi are in correspondence in this case, although quite different in character.

A centrally generated calculus typically has many quotients, i.e. its quantum tangent space L_c itself has further subrepresentations.

Proposition 2.7. *L_c in Proposition 2.6 has a subrepresentation $L_{M_{\mathrm{R}},c} = \mathrm{span}\{x_a | a \in M_{\mathrm{R}}\}$ for any right* Ad*-invariant right ideal $M_{\mathrm{R}} \subseteq \ker\epsilon \subset A$. Hence every non-trivial central element c defines a 'mirror' operation from the moduli space of bicovariant calculi to itself, sending a calculus defined via M_{R} according to a (right-handed version of)* (2) *to the calculus with quantum tangent space $L_{M_{\mathrm{R}},c} \subseteq L_c$.*

Proof. Note first of all that the axioms of a bicovariant calculus are left-right symmetric. The construction $\Gamma = V\otimes A$ in (2) works just as well in a right-handed form $\Gamma = A\otimes V$ were $V = \ker\epsilon/M_{\mathrm{R}}$ and M_{R} is a right ideal stable under $\mathrm{Ad}_{\mathrm{R}}(a) \equiv a_{(2)}\otimes(Sa_{(1)})a_{(3)}$. On the other hand, when acting on a central element, the relations of the quantum double reduce to $ha = \langle h, (Sa_{(1)})a_{(3)}\rangle a_{(2)}$. Hence $h\triangleright(m\triangleright(c-\epsilon(c))) = hm\triangleright(c-\epsilon(c)) = \langle h, (Sm_{(1)})m_{(3)}\rangle m_{(2)}\triangleright(c-\epsilon(c)) \in L_{M_{\mathrm{R}},c}$ for all $m \in M_{\mathrm{R}}$ and $h \in H$, by Ad_{R}-invariance of M_{R}. Also $a\triangleright(m\triangleright(c-\epsilon(c))) = ma\triangleright(c-\epsilon(c)) \in L_{M_{\mathrm{R}},c}$ for all $m \in M_{\mathrm{R}}$ and $a \in A$, as M_{R} is a right ideal. The quantum double action $\triangleright$ is from Lemma 2.1, and $x_a = a\triangleright(c-\epsilon(c))$ as in Proposition 2.6. □

The 'mirror' calculus is the quotient of the universal calculus as in (2) by the (left) ideal

$$\overline{M_{\mathrm{R}}} = \{a \in \ker\epsilon | \langle ma, c\rangle = 0 \quad \forall m \in M_{\mathrm{R}}\}, \tag{14}$$

which is the 'mirror' operation at the level of quotienting ideals. Moreover, the similar 'mirror' operation with left–right interchanged takes us from left ideals to right ideals, and there is a canonical inclusion $M_{\mathrm{R}} \subseteq \overline{\overline{M_{\mathrm{R}}}}$. The calculus with quantum tangent space L_c in Proposition 2.6 is the mirror image of the zero differential calculus, and vice versa.

In addition, the zero differential calculus is the mirror image of the universal differential calculus.

Also clear from this point of view, if L_1, L_2 are subrepresentations of the quantum double then $L_1 \cap L_2$ is also. We denote its calculus by $\Gamma_1 \cdot \Gamma_2$; it is a quotient of both Γ_1, Γ_2. $L_1 + L_2$ is also a subrepresentation and we denote its calculus $\Gamma_1 * \Gamma_2$; it has Γ_1, Γ_2 as quotients. If $L_1 \cap L_2 = \{0\}$ then the resulting calculus is the obvious *direct product* calculus. We say that a differential calculus is *coirreducible* if its corresponding quantum tangent space L is irreducible as a quantum double representation. This implies that the calculus has no proper quotient calculus. Note that this should not be confused with irreducibility for the calculus (no proper subcalculus) which is automatically true for all calculi in the paper as a consequence of axiom (5) in the definition of a bicovariant calculus. Moreover, in nice cases (where the quantum double is 'semisimple' in a suitable sense) one has only to decompose

$$\ker \epsilon = L_1 \oplus L_2 \oplus \cdots \tag{15}$$

into irreducibles in order to classify coirreducible calculi; each distinct irreducible in the decomposition corresponds to an isolated coirreducible calculus and each irreducible with multiplicity typically corresponds to a continuous family of calculi given by a parameter describing the embeddings of the irreducible into its multiple copies in $\ker \epsilon$. Moreover, we see in this situation that the universal differential calculus, which corresponds to $L = \ker \epsilon$, can be built up as a direct product of coirreducible calculi.

3. Calculi on finite groups and enveloping algebras

In this section we apply the formulation of the classification problem in the preceding section to the elementary cases of finite groups and enveloping algebras. The result in the case $A = \mathbb{C}(G)$ (the algebra of functions on a finite group) is more or less known by other means, but recovered now from Proposition 2.2 and with proper attention to irreducibility. We also give the more novel dual case $A = \mathbb{C}G$ (the group algebra). It turns out to be more similar to the quantum group case in Section 4. The classification for classical Lie groups and enveloping algebras remains open, but we make some remarks.

Proposition 3.1. *Let $A = \mathbb{C}(G)$ where G is a finite group, cf. [15]. The coirreducible bicovariant differential calculi are in 1–1 correspondence with the non-trivial conjugacy classes $\mathcal{C} \subset G$. We recover this result from the above approach as corresponding to*

$$L = \mathrm{span}\{x_g \equiv g - e | g \in \mathcal{C}\}, \quad \partial_{x_g} a = a(g \cdot (\ )) - a,$$
$$\Psi^{-1}(a \otimes x_g) = x_g \otimes a(g \cdot (\ ))$$

$$[x_g, x_h] = x_{ghg^{-1}} - x_h, \quad \Psi(x_g \otimes x_h) = x_{ghg^{-1}} \otimes x_g,$$

where e is the group identity element.

Proof. Here $H = \mathbb{C}G$ and we classify all irreducible subspaces $L \subseteq \ker\epsilon \subset \mathbb{C}G$ which are stable under the adjoint action and the action of $\mathbb{C}(G)$ in Lemma 2.1. The algebra $A = \mathbb{C}(G)$ is commutative and elements of the form $x_g = g - e$ are a basis of simultaneous eigenfunctions for its action on $\ker\epsilon$, since $a{\triangleright}x_g = a(g)g - a(e)e - a(g)e + a(e)e = a(g)x_g$ for any $g \in G$. By choosing a a Kronecker delta function we see that if L contains a linear combination involving x_g then it contains x_g itself. Hence $L = \mathrm{span}\{x_g | g \in \mathcal{C}\}$ for some subset $\mathcal{C} \subset G$ not containing e. This is the content of stability under the A part of the quantum double action. The content of stability under the H part of the quantum double action (the adjoint action of G extended linearly) is therefore that $\mathcal{C}$ should be a union of non-trivial conjugacy classes. If L is irreducible as a quantum double module then for any $x_{g_0} \in L$, $L = D(H){\triangleright}x_{g_0} = \mathrm{span}\{h\delta_g{\triangleright}x_{g_0} |\, h, g \in G\} = \mathrm{span}\{x_{hg_0h^{-1}} | h \in G\}$, i.e. $\mathcal{C}$ is exactly one conjugacy class. Conversely, if $\mathcal{C}$ is a non-trivial conjugacy class and $x = \sum_g c_g x_g \in L$ then $D(H){\triangleright}x = \mathrm{span}\{c_g x_{hgh^{-1}} \,|\, h, g \in G\} = L$ as at least one coefficient c_g must be non-zero. The corresponding braided derivations are $\partial_{x_g} a = \langle x_g, a_{(1)}\rangle a_{(2)} = a(g(\ )) - a(e(\ ))$ and the 'quantum Lie bracket' is $[x_g, x_h] = \mathrm{Ad}_g(x_h) - x_h = x_{ghg^{-1}} - x_h$ as stated. Likewise, we compute Ψ from (6) and (8) in the form stated. One also has $\ker\epsilon = \oplus_{\mathcal{C}\neq\{e\}} L_{\mathcal{C}}$, corresponding to the decomposition of $G - \{e\}$ into non-trivial conjugacy classes, i.e. the universal calculus as a direct product of the coirreducibles. □

These calculi are all 'non-classical' in the sense that the braiding needed for the derivation property is non-trivial (when G is non-Abelian). They are in fact a variant of the familiar q-derivative, with q being replaced by a group element taken from the conjugacy class. The non-classical nature also appears as non-commutativity of the calculus in the sense $a\mathrm{d}b \neq (\mathrm{d}b)a$ for some a, b. The calculus is inner type-I with α the characteristic function of $\mathcal{C} \cup \{e\}$, and inner type-II with α the characteristic function of $\mathcal{C}$. We can also apply our formalism to $A = \mathbb{C}G$. If G is Abelian, we have $\mathbb{C}G = \mathbb{C}(\hat{G})$ and return to the preceding example applied to the dual group. But when G is non-Abelian, the algebra A is non-commutative and we are really doing 'non-commutative geometry'.

Proposition 3.2. *Let $A = \mathbb{C}G$ where G is a finite group. The coirreducible bicovariant differential calculi are in 1–1 correspondence with pairs V, λ, where V is a non-trivial irreducible (right) representation of G and $\lambda \in P(V^*)$. The corresponding calculus has dimension* $\dim V$ *and*

$$L = \mathrm{span}\{x_v \equiv \langle v{\triangleleft}(\ ), \lambda\rangle - \langle v, \lambda\rangle 1 \mid v \in V\},$$
$$\partial_{x_v}(g) = (\langle v{\triangleleft}g, \lambda\rangle - \langle v, \lambda\rangle)g, \quad \Psi^{-1}(g \otimes x_v) = x_{v\triangleleft g} \otimes g$$

and trivial 'quantum Lie bracket'.

Proof. Here $H = \mathbb{C}(G)$ is commutative. Hence the adjoint action in Lemma 2.1 is trivial (as is the bracket [,] and its associated braiding). We therefore need only to classify irreducible subspaces $L \subseteq \ker\epsilon \subset \mathbb{C}(G)$ under the action of $\mathbb{C}G^{\mathrm{op}}$. This action is $h{\triangleright}x = x(h(\ )) - x(h)1$ for all $x \in \ker\epsilon$, which is the standard projection Π to $\ker\epsilon$ of the right regular representation of G on $\mathbb{C}(G)$ by multiplication from the left in the group. The

Peter–Weyl decomposition $\mathbb{C}(G)\cong\mathbb{C}\oplus_{V\neq\mathbb{C}} V\otimes V^*$ projected via the projection Π is an isomorphism $\oplus_{V\neq\mathbb{C}} V\otimes V^*\cong\ker\epsilon$, giving the decomposition of this into irreducibles. In the Peter–Weyl decomposition, the element $v\otimes\lambda$ maps to the function $\langle v\triangleleft(\ ),\lambda\rangle\in\mathbb{C}(G)$, giving the form of L shown. We need $\lambda\neq 0$ and we identify all λ which are related by a phase since these give the same L, i.e. the continuous parameter is $\lambda\in P(V^*)=\mathbb{C}P^{\dim V-1}$. The braided-derivation is $\partial_{x_v}g=\langle x_v,g\rangle g$ on group-like elements of $\mathbb{C}G$, which gives the form shown. The group-like elements are simultaneous eigenfunctions for all the braided-derivations. The braiding is easily computed as $\Psi^{-1}(g\otimes x_v)=g\triangleright x_v\otimes g=\langle v\triangleleft g(\ ),\lambda\rangle\otimes g-\langle v\triangleleft g,\lambda\rangle\otimes g=x_{v\triangleleft g}\otimes g$. □

Note that a basis of V^* specifies $\dim V$ isomorphic copies of V in the Peter–Weyl decomposition. However, we need here not only the multiplicities but the actual irreducible subspaces $L\subset\ker\epsilon$. We obtain a subspace isomorphic to V for every non-trivial linear combination (modulo an overall scale) of the basis elements, i.e. a continuous family of calculi parametrised by the projective space $P(V^*)$ for each irreducible representation V. Also, since irreducible representations of G correspond to characters, one can recast this result in terms of these. For a given character χ we identify V^*_χ as the quotient of $\mathbb{C}G$ where $[\lambda]=[\lambda']$ if $\chi(g\lambda)=\chi(g\lambda')$ for all g. Then coirreducible calculi are in 1–1 correspondence with pairs $\chi,[\lambda]$ according to

$$L=\mathrm{span}\{x_g\equiv\chi(g(\ )\lambda)-\chi(g\lambda)1\,|\,g\in G\}$$
$$\partial_{x_g}h=(\chi(gh\lambda)-\chi(g\lambda))h,\qquad \Psi^{-1}(h\otimes x_g)=x_{gh}\otimes h,$$

where $g,h\in G$. Here V_χ is the vector space spanned by $\chi((\ )g)$ as g runs over G and is an irreducible (right) representation of G acting by left multiplication in the argument of χ. From this, it is clear that these calculi on $\mathbb{C}G$ are all centrally generated as $\tilde{L}_c$ by $c=\chi((\ )\lambda)$.

Finally, we consider the differential calculi on a classical Lie group coordinate ring $A=\mathbb{C}(G)$. Here $\mathbb{C}(G)$ denotes an algebraic model of the functions on G constructed as a Hopf algebra non-degenerately paired to the enveloping algebra $U(g)$, see [7]. Quantum tangent spaces $L\subset U(g)$ themselves can viewed as crossed $U(g)$-modules independently of $\mathbb{C}(G)$.

Proposition 3.3. *Let g be a Lie algebra. For each natural number n there is a bicovariant quantum tangent space $L=g+gg+\cdots+g^n$, the subspace of $U(g)$ of degree $\leq n$ and ≥ 1. For example, for $n=2$:*

$$L=\mathrm{span}\{\xi,\eta\zeta\,|\,\xi,\eta,\zeta\in g\},\quad \partial_\xi=-\tilde{\xi},\quad \partial_{\eta\zeta}=\tilde{\zeta}\tilde{\eta},$$
$$\Psi^{-1}(a\otimes\xi)=\xi\otimes a,\quad \Psi^{-1}(a\otimes\eta\zeta)=\eta\zeta\otimes a-\zeta\otimes\tilde{\eta}(a)-\eta\otimes\tilde{\zeta}(a),$$
$$[\xi,x]=\xi x-x\xi,\quad [\eta\zeta,x]=\eta\zeta x-\eta x\zeta-\zeta x\eta+x\zeta\eta,$$
$$\Psi(\xi\otimes x)=x\otimes\xi,\quad \Psi(\xi\eta\otimes x)=[\xi,x]\otimes\eta+[\eta,x]\otimes\xi+x\otimes\xi\eta$$

for all $x\in L$. Here $\tilde{\xi}$ is the right-invariant vector-field associated to $\xi\in g$.

Proof. Note that the degree of a given element in $U(g)$ is not well-defined but the subspace spanned by products of up to n elements is. We show only that such a subspace $L^{(n)}$ forms a quantum double subrepresentation. To see that it is closed under the adjoint action of $U(g)$ it suffices to see that it is closed under the action of each $\xi \in g$. This action is by commutator in $U(g)$. Hence assuming the result for $L^{(n-1)}$ and the Leibniz rule for commutators, we obtain the result for $L^{(n)}$ by induction. The other part of the quantum double action (that of $A = \mathbb{C}(G)$) is given by evaluation against the left coaction $\beta = \Delta - \mathrm{id}\otimes 1$. Then $\beta(\xi x) = \Delta(\xi x) - \xi x\otimes 1 = (\xi\otimes 1)\Delta x + (1\otimes\xi)\Delta x - \xi x\otimes 1 = (\xi\otimes 1)\beta(x) + (1\otimes\xi)\beta(x) + x\otimes\xi \in U(g)\otimes L^{(n)}$ as $\beta(x) \in U(g)\otimes L^{(n-1)}$ and $n \geq 1$. Here $x \in L^{(n-1)}$ and we proceed by induction. The explicit computations for $L^{(2)}$ are immediate from the form of the coproduct on $\eta\zeta$ in the formulae above. Here, $\partial_\xi(a) = \langle\xi, a_{(1)}\rangle a_{(2)} = (\mathrm{d}/\mathrm{d}t)|_0 a(\mathrm{e}^{t\xi}(\ )) = -\tilde{\xi}(a)$ for $a \in \mathbb{C}(G)$. In the matrix Lie algebra case, this is given by the fundamental representation used in defining the pairing between $U(g)$ and $\mathbb{C}(G)$, i.e. it is actually algebraic. □

We see that it is possible to view higher order differential operators as if they are 'first-order vector fields' – but braided. A second-order operator, for example, is clearly not a derivation in the usual sense but it is a braided-derivation for suitable Ψ. For example, one could compute its 'flow' as a corresponding braided-exponential. This opens up the possibility of a 'geometrical' picture for the evolution of quantum systems generated by second- or higher-order Hamiltonians, to be given in detail elsewhere.

On the other hand, we do not attempt to classify all bicovariant calculi here. This would appear to be an interesting problem in the classical theory of enveloping algebras: find all subspaces L which are stable under the adjoint action and under the left coaction $\beta = \Delta - \mathrm{id}\otimes 1$. Moreover, the $L^{(n)}$ are of course not coirreducible. Instead, we have a filtration

$$g = L^{(1)} \subset L^{(2)} \subset L^{(3)} \cdots \subset L^{(\infty)} = \ker\epsilon, \tag{16}$$

where $g = L^{(1)}$ corresponds to the classical differential calculus on $\mathbb{C}(G)$. At the level of bicovariant calculi we have a sequence of quotients of the universal one (of all finite degree invariant differential operators) eventually quotienting down to the standard one. The Casimir construction provides further bicovariant calculi falling in between these $L^{(n)}$.

Proposition 3.4. *Let g be a Lie algebra and $c = K^{(1)}K^{(2)}$ the quadratic Casimir associated to symmetric* ad*-invariant $K = K^{(1)}\otimes K^{(2)} \in g\otimes g$ (summation understood). The extension $g \oplus \mathbb{C}$ by c is a quantum tangent space. g acts by right-invariant vector fields and c acts as a second-order operator viewed as a braided-vector field. The quantum Lie bracket restricted to g is its usual Lie bracket. The other cases and the braiding are*

$$[\xi, c] = 0, \quad [c, c] = 0, \quad [c, \xi] = [K^{(1)}, [K^{(2)}, \xi]], \quad \Psi(\xi\otimes\eta) = \eta\otimes\xi$$
$$\Psi(\xi\otimes c) = c\otimes\xi, \quad \Psi(c\otimes\xi) = \xi\otimes c + 2[K^{(1)}, \xi]\otimes K^{(2)},$$
$$\Psi(c\otimes c) = c\otimes c$$

for all $\xi \in g$. In the matrix Lie algebra case, if K is non-degenerate then $g = L_c$ and $g \oplus \mathbb{C} = \tilde{L}_c$ are centrally generated via Proposition 2.6.

Proof. The first part is the restriction of $g + g^2$ in Proposition 3.3 to $g \oplus \mathbb{C}$. In particular, $\Psi(c\otimes\xi) = [K^{(1)}, \xi]\otimes K^{(2)} + [K^{(2)}, \xi]\otimes K^{(1)} + \xi\otimes c$ is as stated by symmetry of K. For the second part we assume (for simplicity) that $\rho : g \subseteq M_N$ (the $N \times N$ matrices) is an inclusion of the Lie algebra and take $A = \mathbb{C}(G)$ generated by corresponding matrix elements t^i_j with pairing $\langle t^i{}_j, \xi\rangle = \rho(\xi)^i{}_j$. For $\tilde{L}_c$ in Proposition 2.6, we have $x_a = \langle a, c_{(1)}\rangle c_{(2)} - \langle a, c\rangle 1 = \epsilon(a)c + 2\langle a, K^{(1)}\rangle K^{(2)}$. Hence, in particular, $x_{t^i_j} = \delta^i{}_j c + 2\rho(K^{(1)})^i{}_j K^{(2)}$. Since ρ is faithful, its dual $M_N^* \to g^*$ is surjective. If we assume further that K is non-degenerate when viewed as a map $K : g^* \to g$ then we see that $\mathrm{span}\{x_{t^i_j}\} = g \oplus \mathbb{C}$. Moreover, $x_{t^i{}_j a} = a\triangleright x_{t^i_j} = \delta^i{}_j x_a + \epsilon(a)2\rho(K^{(1)})^i{}_j K^{(2)} = \delta^i{}_j(x_a - \epsilon(a)c) + \epsilon(a)x_{t^i_j}$, using the action $\triangleright$ from Lemma 2.1. Here $a\triangleright\xi = \langle a, \xi_{(1)}\rangle\xi_{(2)} - \langle a, \xi\rangle 1 = \epsilon(a)\xi$ for all $\xi \in g$. Hence, by induction, all $x_a \in g \oplus \mathbb{C}$ as required. For L_c we exclude the diagonals $i = j$ so that the span is g alone. Note that when g is simple we have a unique K and a unique quadratic Casimir c. □

Equivalently, the mirror operation in Proposition 2.7 in this case turns the zero differential calculus into the classical one, and vice versa. Also, if g under the adjoint action is isotypical (as for sl_2) then the quantum Lie bracket $[c, \xi]$ here is fixed multiple of ξ. The simplest case $L = sl_2 \oplus \mathbb{C}$ corresponds to the non-standard four-dimensional differential calculus on $SU(2)$ which has been studied in [15] as the $q \to 1$ limit of the known four-dimensional calculus on the quantum group $SU_q(2)$ in [2]. Similarly, $L^{(n-1)} \oplus \mathbb{C}$ corresponds to a natural calculus in between the calculi corresponding to $L^{(n-1)}$ and $L^{(n)}$, whenever we have a degree n central element. Intermediate calculi are generally what arise when we take the limit of quantum group differential calculi (these will be classified in Section 4), i.e. this is a general feature. Put another way, we will see from the classification in the next section that the standard $\dim g$-dimensional calculus on a simple Lie group G violates the 'principle of q-deformisability'; only certain extensions of ordinary vector fields on a Lie group by higher order vector fields can deform to calculi on G_q.

4. Calculi on factorisable quantum groups

In this section we present our main result, which is a classification of the bicovariant calculi for a certain natural class of quantum groups. We then discuss the application of the result to the standard quantum groups G_q.

We recall that a 'strict quantum group' or quasitriangular Hopf algebra is factorisable [16] if $\mathcal{R}_{21}\mathcal{R}$ viewed as a map $\mathcal{Q} : A \to H$ by $\mathcal{Q}(a) = (a\otimes \mathrm{id})(\mathcal{R}_{21}\mathcal{R})$ is an isomorphism. This is the strongest form; one may also demand separately that the map is injective or surjective. We also require a kind of 'semisimplicity' condition in the sense that there is a Peter–Weyl decomposition

$$\oplus_V V^*\otimes V \cong A \tag{17}$$

provided by the matrix elements of the inequivalent finite-dimensional irreducible representations V of H. If V is such a representation, with basis $\{e_i\}$ and dual basis $\{f^i\}$, we define the matrix elements $\rho^i{}_j \in A$ by $h \triangleright e_i = e_j \rho^j{}_i(h)$ and the above map by $f^i \otimes e_j \mapsto \rho^i{}_j$.

Lemma 4.1. *Let H be a factorisable quantum group with dual A. The map $\mathcal{Q}$ identifies* $\ker \epsilon \subset A$ *and* $\ker \epsilon \subset H$. *Under this identification, the action of the quantum double in Lemma* 2.1 *becomes the action on* $\ker \epsilon \subset A$ *given by*

$$h \triangleright a = a_{(2)} \langle h, (Sa_{(1)}) a_{(3)} \rangle$$
$$b \triangleright a = \langle b, \mathcal{R}'^{(1)} \mathcal{R}^{(2)} \rangle \langle a_{(1)}, \mathcal{R}'^{(2)} \rangle \langle a_{(3)}, \mathcal{R}^{(1)} \rangle a_{(2)} - \langle b, \mathcal{Q}(a) \rangle 1$$

for all $h \in H$ and $b \in A$. Here, $\mathcal{R}' \equiv \mathcal{R}'^{(1)} \otimes \mathcal{R}'^{(2)}$ is a second copy of $\mathcal{R}$.

Proof. It is immediate from the counity property of the quasitriangular structure $\mathcal{R}$ that $\mathcal{Q}(1) = 1$. Hence $\mathcal{Q}(\ker \epsilon) = \ker \epsilon \subset H$. Moreover, we know from Ad-invariance of the quantum Killing form that $\mathrm{Ad}_h \circ \mathcal{Q}(a) = \mathcal{Q}(\mathrm{Ad}^*_h a)$ where Ad^*_h is the left quantum coadjoint action as stated for $h \triangleright a$ in the lemma, and Ad is the quantum adjoint action used for x in Lemma 2.1. For a proof see [9] or the text [7]. The new part concerns the other action:

$$\begin{aligned} b \triangleright \mathcal{Q}(a) &= \langle b, \mathcal{Q}(a)_{(1)} \rangle \mathcal{Q}(a)_{(2)} - \langle b, \mathcal{Q}(a) \rangle 1 \\ &= \langle a, \mathcal{R}_1^{(2)} \mathcal{R}_2^{(1)} \rangle \langle b, \mathcal{R}_1^{(1)}{}_{(1)} \mathcal{R}_2^{(2)}{}_{(1)} \rangle \mathcal{R}_1^{(1)}{}_{(2)} \mathcal{R}_2^{(2)}{}_{(2)} - \langle b, \mathcal{Q}(a) \rangle \mathcal{Q}(1) \\ &= \langle a, \mathcal{R}_1^{(2)} \mathcal{R}_3^{(2)} \mathcal{R}_4^{(1)} \mathcal{R}_2^{(1)} \rangle \langle b, \mathcal{R}_1^{(1)} \mathcal{R}_2^{(2)} \rangle \mathcal{R}_3^{(1)} \mathcal{R}_4^{(2)} - \langle b, \mathcal{Q}(a) \rangle \mathcal{Q}(1) \\ &= \langle a_{(1)}, \mathcal{R}_1^{(2)} \rangle \langle a_{(3)}, \mathcal{R}_2^{(1)} \rangle \langle b, \mathcal{R}_1^{(1)} \mathcal{R}_2^{(2)} \rangle \mathcal{Q}(a_{(2)}) - \langle b, \mathcal{Q}(a) \rangle \mathcal{Q}(1) = \mathcal{Q}(b \triangleright a) \end{aligned}$$

for all $a \in \ker \epsilon \subset A$ and $b \in A$, where $\mathcal{R}_1^{(1)} \otimes \mathcal{R}_1^{(2)}, \ldots, \mathcal{R}_4^{(1)} \otimes \mathcal{R}_4^{(2)}$ are four copies of $\mathcal{R}$. The first equality is the action of A in Lemma 2.1. The second puts in the formula for $\mathcal{Q}$. The third is the coproduct property of $\mathcal{R}$ and finally we recognise the required result in terms of the action $b \triangleright a$ stated. Hence $\mathcal{Q}$ intertwines the stated action of the quantum double with the action in Lemma 2.1. Note that this computation also works at the level of a coaction of H rather than an action by $b \in A$ (i.e. the action of the quantum double remains A-regular). □

So the possible quantum tangent spaces L are in 1–1 correspondence with subrepresentations of $\ker \epsilon \subset A$ under this action of the quantum double. This action looks more complicated than before. However, there is a well-known isomorphism in the factorisable case of the quantum double with $H \bowtie H$. The latter is $H \otimes H$ as an algebra and has a coalgebra which is a twisting of the tensor product one. The map θ to $H \bowtie H$ is [16]

$$\theta(h \otimes a) = h_{(1)} \mathcal{R}^{-(2)} \otimes h_{(2)} \mathcal{R}^{(1)} \langle \mathcal{R}^{-(1)} \mathcal{R}^{(2)}, a \rangle. \tag{18}$$

The full details of the isomorphism and an explicit formula for θ^{-1} are in the author's text [7].

Proposition 4.2. *The action in Lemma* 4.1 *of the quantum double, in the form* $H\bowtie H$ *acting on* $\ker\epsilon \subset A$, *takes the form*

$$(h\otimes 1)\triangleright a = \langle Sh, a_{(1)}\rangle a_{(2)} - 1\langle Sh, a\rangle, \quad (1\otimes g)\triangleright a = a_{(1)}\langle g, a_{(2)}\rangle - 1\langle g, a\rangle$$

for all $h, g \in H$ *and* $a \in \ker\epsilon \subset A$.

Proof. To find the action of $H\bowtie H$ we need the explicit inversion formula for θ in [7]. Then $(h\otimes 1)\triangleright a = \theta^{-1}(h\otimes 1)\triangleright a$, etc., can be computed, and one obtains the result stated in the proposition. Once these actions have been obtained, however, it is enough (and rather easy) to verify that pull back along θ indeed recovers the action of $H\bowtie A^{\mathrm{op}}$ in Lemma 4.1. Thus,

$$\begin{aligned}
\theta(h\otimes 1)\triangleright a &= (h_{(1)}\otimes h_{(2)})\triangleright a = (h_{(1)}\otimes 1)\triangleright a_{(1)}\langle h_{(2)}, a_{(2)}\rangle - (h_{(1)}\otimes 1)\triangleright 1\langle h_{(2)}, a\rangle \\
&= \langle Sh_{(1)}, a_{(1)}\rangle a_{(2)}\langle h_{(2)}, a_{(3)}\rangle - \langle Sh_{(1)}, a_{(1)}\rangle 1\langle h_{(2)}, a_{(2)}\rangle \\
&= \langle h, (Sa_{(1)})a_{(3)}\rangle a_{(2)} \\
\theta(1\otimes b)\triangleright a &= (\mathcal{R}^{-(2)}\otimes\mathcal{R}^{(1)})\triangleright a\langle\mathcal{R}^{-(1)}\mathcal{R}^{(2)}, b\rangle \\
&= (\mathcal{R}^{-(2)}\otimes 1)\triangleright a_{(1)}\langle\mathcal{R}^{(1)}, a_{(2)}\rangle\langle\mathcal{R}^{-(1)}\mathcal{R}^{(2)}, b\rangle \\
&\quad - (\mathcal{R}^{-(2)}\otimes 1)\triangleright 1\langle\mathcal{R}^{(1)}, a\rangle\langle\mathcal{R}^{-(1)}\mathcal{R}^{(2)}, b\rangle \\
&= \langle S\mathcal{R}^{-(2)}, a_{(1)}\rangle a_{(2)}\langle\mathcal{R}^{(1)}, a_{(3)}\rangle\langle\mathcal{R}^{-(1)}\mathcal{R}^{(2)}, b\rangle \\
&\quad - \langle S\mathcal{R}^{-(2)}, a_{(1)}\rangle 1\langle\mathcal{R}^{(1)}, a_{(2)}\rangle\langle\mathcal{R}^{-(1)}\mathcal{R}^{(2)}, b\rangle \\
&= a_{(2)}\langle\mathcal{R}'^{(2)}, a_{(1)}\rangle\langle\mathcal{R}^{(1)}, a_{(3)}\rangle\langle\mathcal{R}'^{(1)}\mathcal{R}^{(2)}, b\rangle - \langle\mathcal{Q}(a), b\rangle 1
\end{aligned}$$

as required. We used the form of θ, the actions as stated in the proposition and, in the last line, the antipode property $\mathrm{id}\otimes S\mathcal{R}^{-1} = \mathcal{R}$ of a quasitriangular structure. Our notation is $\mathcal{R}^{-(2)}\otimes\mathcal{R}^{-(1)} = \mathcal{R}^{-1}$. □

So, quantum tangent spaces L are in correspondence with subrepresentations of $\ker\epsilon$ under this action of $H\otimes H$. We can now obtain our main result.

Theorem 4.3. *Let* H *be a factorisable quantum group with dual* A, *and suppose that the Peter–Weyl decomposition* (17) *holds. Then the finite dimensional bicovariant coirreducible calculi on* A *are in* 1–1 *correspondence with the non-trivial finite-dimensional irredicible representations* V *of* H. *The corresponding calculus has dimension* $(\dim V)^2$ *and*

$$\begin{aligned}
&L = \mathrm{span}\{x^i{}_j \equiv \mathcal{Q}(\rho^i{}_j - 1\delta^i{}_j)|\, i, j = 1, \dots, \dim V\}, \\
&\partial_{x^i{}_j}(a) = \mathcal{Q}(\rho^i{}_j\otimes a_{(1)})a_{(2)} - \delta^i{}_j a, \\
&\qquad \Psi^{-1}(a\otimes x^i{}_j) = x^a{}_b\otimes a_{(3)}\mathcal{R}(a_{(1)}\otimes\rho^i{}_a)\mathcal{R}(\rho^b{}_j\otimes a_{(2)}), \\
&[x^i{}_j, x^k{}_l] = x^a{}_b\mathcal{Q}(\rho^i{}_j\otimes(S\rho^k{}_a)\rho^b{}_l) - x^k{}_l\delta^i{}_j, \\
&\Psi(x^i{}_j\otimes x^k{}_l) = x^m{}_n\otimes x^a{}_b\mathcal{R}((S\rho^c{}_m)\rho^n{}_d\otimes\rho^i{}_a)\mathcal{R}(\rho^b{}_j\otimes(S\rho^k{}_c)\rho^d{}_l),
\end{aligned}$$

where we also regard the quantum Killing form and quasitriangular structure as functionals $\mathcal{Q}, \mathcal{R} : A\otimes A \to \mathbb{C}$

Proof. We first separate off the trivial representation in (17), so $A \cong \mathbb{C} \oplus (\oplus_{V \neq \mathbb{C}} V^* \otimes V)$ where the sum is over non-trivial V. The projection $\Pi(a) = a - 1\epsilon(a)$ from $A \to \ker\epsilon$ establishes an isomorphism

$$\ker\epsilon \cong \oplus_{V \neq \mathbb{C}} V^* \otimes V. \tag{19}$$

This is because Π and the projection to $\oplus_{V \neq \mathbb{C}} V^* \otimes V$ have the same kernel, namely the span of the identity element in A. By Proposition 4.2, we therefore have an isomorphism of $H \otimes H$ modules, where the second H acts on V as in the Peter–Weyl decomposition (the given irreducible representation V) and the first copy of H acts on V^* by the conjugate representation $h \triangleright f = f(Sh \triangleright (\ ))$ for $f \in V^*$. Next, as $H \otimes H$ modules, these $V^* \otimes V$ are distinct and irreducible. Hence they are precisely the choices for irreducible subrepresentations of $\ker\epsilon \subset A$.

The explicit formula for the braided-derivations and their requisite braiding are easily computed from the formulae in Proposition 2.3. From the proof of Lemma 4.1 we have

$$\begin{aligned}(\Delta - \mathrm{id}\otimes 1)x^i{}_j &= \mathcal{R}^{(1)}\mathcal{R}'^{(2)}\langle(\rho^i{}_j - \delta^i{}_j)_{(1)}, \mathcal{R}^{(2)}\rangle\langle(\rho^i{}_j - \delta^i{}_j)_{(3)}, \mathcal{R}'^{(1)}\rangle\\ &\quad \otimes \mathcal{Q}((\rho^i{}_j - \delta^i{}_j)_{(2)}) - \mathcal{Q}(\rho^i{}_j - \delta^i{}_j)\otimes 1\\ &= \mathcal{R}^{(1)}\mathcal{R}'^{(2)} \otimes \langle\rho^i{}_a, \mathcal{R}^{(2)}\rangle\langle\rho^b{}_j, \mathcal{R}'^{(1)}\rangle\mathcal{Q}(\rho^a{}_b) - \mathcal{Q}(\rho^i{}_j)\otimes 1\\ &= \mathcal{R}^{(1)}\mathcal{R}'^{(2)} \otimes \langle\rho^i{}_a, \mathcal{R}^{(2)}\rangle\langle\rho^b{}_j, \mathcal{R}'^{(1)}\rangle x^a{}_b.\end{aligned}$$

Evaluation against this is the action of A in Lemma 4.1, which is the action needed to compute the braiding. Thus, $\Psi^{-1}(a \otimes x^i{}_j) = a_{(2)} \otimes \langle a_{(1)}, \mathcal{R}^{(1)}\mathcal{R}'^{(2)}\rangle\langle\rho^i{}_a, \mathcal{R}^{(2)}\rangle\langle\rho^b{}_j, \mathcal{R}'^{(1)}\rangle x^a{}_b$, which can be written in the form shown where $\mathcal{R}$ is regarded as a functional on $A \otimes A$. The quantum Lie bracket and its braiding from Proposition 2.4 are also easily computed and follow the same lines as in [9,17], except that we are not tied to any particular representation V or any fixed R-matrix; we include the proofs only for completeness in our present conventions. Thus, by Ad-invariance of $\mathcal{Q}$ we have

$$\begin{aligned}[x^i_j, x^k_l] &= \mathcal{Q}(\mathcal{Q}(\rho^i_j - \delta^i_j)\triangleright(\rho^k{}_l - \delta^k{}_l))\\ &= \mathcal{Q}(\rho^a{}_b)\langle\mathcal{Q}(\rho^i{}_j), (S\rho^k{}_a)\rho^b{}_l\rangle - \delta^i{}_j\mathcal{Q}(\rho^k{}_l)\\ &= x^a{}_b\langle\mathcal{Q}(\rho^i{}_j), (S\rho^k{}_a)\rho^b{}_l\rangle - \delta^i{}_j x^k{}_l,\end{aligned}$$

which we write in the form stated where $\mathcal{Q} = \mathcal{R}_{21}\mathcal{R}$ is regarded as a functional on $A \otimes A$. Here $\triangleright$ is the quantum coadjoint action of H in Lemma 4.1. Finally, using the above result for $\Delta x^i{}_j$ and Ad-invariance of $\mathcal{Q}$, we have

$$\begin{aligned}\Psi(x^i{}_j \otimes x^k{}_l) &= [x^i{}_{j(1)}, x^k{}_l] \otimes x^i{}_{j(2)} - [x^i{}_j, x^k{}_l] \otimes 1\\ &= \mathcal{Q}(\mathcal{R}^{(1)}\mathcal{R}'^{(2)} \triangleright (\rho^k{}_l - \delta^k{}_l)) \otimes \langle\rho^i{}_a, \mathcal{R}^{(2)}\rangle\langle\rho^b{}_j, \mathcal{R}'^{(1)}\rangle x^a{}_b\\ &= \mathcal{Q}(\rho^c{}_d) \otimes x^a{}_b\langle\mathcal{R}^{(1)}\mathcal{R}'^{(2)}, (S\rho^k{}_c)\rho^d{}_l\rangle\langle\rho^i{}_a, \mathcal{R}^{(2)}\rangle\langle\rho^b{}_j, \mathcal{R}'^{(1)}\rangle\\ &\quad - \delta^k{}_l \otimes x^i{}_j\\ &= x^c{}_d \otimes x^a{}_b\langle\mathcal{R}^{(1)}\mathcal{R}'^{(2)}, (S\rho^k{}_c)\rho^d{}_l\rangle\langle\rho^i{}_a, \mathcal{R}^{(2)}\rangle\langle\rho^b{}_j, \mathcal{R}'^{(1)}\rangle,\end{aligned}$$

which we write in the form stated. Note that both the expressions $\mathcal{Q}(\rho^i{}_j\otimes(S\rho^k{}_a)\rho^b{}_l)$ and $\mathcal{R}((S\rho^c{}_m)\rho^n{}_d\otimes\rho^i{}_a)\mathcal{R}(\rho^b{}_j\otimes(S\rho^k{}_c)\rho^d{}_l)$ can be expanded as four-fold products of the matrices $R = (\rho\otimes\rho)\mathcal{R}$, its inverse and $\tilde{R} = (\rho\otimes\rho\circ S)\mathcal{R}$. This step and the resulting R-matrix formulae are identical in form to the computation of the quantum Lie algebra 'structure constants' in [9] and the quadratic relations of the braided matrices in [17] (the matrix denoted Ψ' there), respectively. Hence we omit the proofs and note only that, after rearranging the R-matrices, one has the same form as for a quantum or braided-Lie algebra of matrix type, namely

$$R_{21}[x_1, Rx_2] = x_2Q - Qx_2, \quad R_{21}\Psi(x_1\otimes Rx_2) = x_2R_{21}\otimes x_1R, \tag{20}$$

where the numerical suffices denote positions in a matrix tensor product and $Q = R_{21}R$. The relation between (20), braided matrices $\mathbf{u} = x + \mathrm{id}$ and the quantum double is explained further in [8] (where the quantum double braiding Ψ is denoted $\check{\mathbf{R}}$). On the other hand, now (20) applies to any irreducible representation V of H and not some fundamental basic representation, which need not exist. □

Let us note that if $\mathcal{R}$ is a quasitriangular structure in a quantum group then so is $\mathcal{R}_{21}^{-1}$. Thus all results involving a quasitriangular Hopf algebra have a 'conjugate' one in which this conjugate $\mathcal{R}_{21}^{-1}$ is used instead of $\mathcal{R}$. This conjugation is also intimately tied to the $*$-operation or complex conjugation in many systems [18]. In the above theorem, we see that for every V we have equally well the conjugate

$$\bar{L} = \mathrm{span}\{\bar{x}^i{}_j \equiv \bar{\mathcal{Q}}(\rho^i{}_j - 1\delta^i{}_j) |\, i, j = 1, \ldots, \dim V\}, \tag{21}$$

where $\bar{\mathcal{Q}}(a) = (a\otimes\mathrm{id})(\mathcal{R}^{-1}\mathcal{R}_{21}^{-1})$. Here $\bar{L}$ is isomorphic to L but the isomorphism (which is $\bar{\mathcal{Q}}\circ\mathcal{Q}^{-1}$ restricted to L) is non-trivial. This fits also with the general point of view of quasi-$*$ structures on inhomogeneous quantum groups [18] where the tensor product of unitaries is unitary only up to a non-trivial isomorphism.

These results can be applied formally to the standard quantum groups $H = U_q(g)$ with dual $A = G_q$ associated to complex semisimple Lie algebras, provided we work over formal power-series $\mathbb{C}[[\hbar]]$ and introduce suitable logarithms for some of the G_q generators, etc. Or, if we want to work algebraically over $\mathbb{C}$ (with generic q), we need to localise and introduce roots of some of the generators of G_q and use the algebraic form of $U_q(g)$ where $q^{H/2}$, etc., is regarded as a single generator. This is clear from the standard cases such $SU_q(2)$: In standard notations the value of $\mathcal{Q}$ on the generators is

$$\mathcal{Q}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^H & q^{-1/2}(q-q^{-1})q^{H/2}X_- \\ q^{-1/2}(q-q^{-1})X_+q^{H/2} & q^{-1}C - q^{-2}q^H \end{pmatrix}, \tag{22}$$

where $C = q^{H-1} + q^{-H+1} + (q-q^{-1})^2X_+X_-$ is the q-quadratic Casimir. According to [16], the standard quantum groups are all factorisable modulo such formal extensions. Likewise, the Peter–Weyl decomposition (17) holds formally for the standard semisimple g. This is because the category of finite-dimensional representations in the classical and quantum cases are generically equivalent, and the assumption holds in some form for the classical case. Note also that the entries in (22) projected to $\ker\epsilon$ span a four-dimensional L

associated to the spin 1/2 representation, and has the structure in Theorem 4.3 without any power-series. Indeed, the quantum double of $U_q(su_2)$ is known to be a q-deformation of the Lorentz group and hence the lowest possible generic representation is the four-dimensional one on q-Minkowski space. In this simplest case, $\bar{L}$ is the same subspace L. The latter also coincides with L_C from Proposition 2.6 with C the q-quadratic Casimir above, and is a subspace of $L_{\alpha,1}$ from Proposition 2.5 with $\alpha = (qa + q^{-1}d)/q^{-2}(q^3 - 1)(q - 1)$ the normalised q-trace.

Therefore, we should understand Theorem 4.3 not as a complete algebraic classification for a given version of each given G_q (this is a much harder problem and has been addressed so far [3] only for variants of the standard calculi of low dimension), but as a classification of those calculi which are 'generic' in the sense that they extend to the various localisations and square roots of the generators, etc. needed for exact factorisability. In other words, there are natural calculi, corresponding to L (or $\bar{L}$) for each finite-dimensional irreducible representation V, and these are all modulo 'pathological' possibilities for particular q for particular versions of particular G_q.

For the A, B, C, D series we have a natural 'fundamental representation' V and in this case it should be clear that the calculus corresponding to L is the one found by Jurco [13] by other means. Here $\rho = \mathbf{t}$, the generator of G_q, and $x^i{}_j = (l^+Sl^-)^i{}_j - \delta^i{}_j$ in Theorem 4.3. This calculus is also of the form L_c for the central element $c = \mathrm{Tr}_q l^+Sl^-$, where Tr_q is the general q-trace. Indeed, $\Delta(l^+Sl^-)^i{}_j = l^{+i}{}_a Sl^{-b}{}_j \otimes (l^+Sl^-)^a{}_b$, from which it is immediate that $x_{t^i{}_j} \equiv t^i{}_j \triangleright c$ and hence $x_{t^i{}_j a} \equiv a \triangleright x_{t^i{}_j}$ in Proposition 2.6 are linear combinations of $x^i{}_j$ for all $a \in G_q$. Irreducibility then implies that the calculi coincide. We therefore have new constructions for these standard calculi and the result that their generalisation to other irreducible representations exhausts all the generic first-order bicovariant differential calculi on the standard semisimple quantisations.

5. Concluding remarks

We conclude with some remarks about further work. First, from the first order '1-forms' one may naturally construct a whole exterior algebra [2], forming a super-Hopf algebra [19]. But other constructions of the exterior algebra may also be possible and should be classified.

Second, the results in Section 2 have an analogue for braided groups. These are needed to include q-deformations $\mathbb{R}^n_q$ and $\mathbb{R}^{1,3}_q$, etc., with their additive (braided) coproduct. The classification of differential calculi on such objects would therefore seem to be the starting point for some form of q-geometry based on $\mathbb{R}^n$. Our result in this direction is that generically there is only one coirreducible braided-bicovariant differential calculus on $\mathbb{R}^{1,3}_q$ (say), and it is infinite-dimensional. Its braided tangent space $\mathcal{L}$ consists (in a suitable completion) of a q-deformation of the space of solutions of the massless Klein–Gordon equation projected to the functions vanishing at the origin. Briefly, the sketch is as follows. Let B be a braided group in a braided category generated by 'background quantum group' H as its category of modules. We define a braided-bicovariant calculus Γ in the obvious way and proceed in a similar manner to Section 2. The role of the quantum double is now played by the 'double-bosonisation' $B^{*}\bowtie H \bowtie B$ quantum group [20]. This acts on $\ker\epsilon \subset B$ and the possible braided tangent spaces $\mathcal{L}$ are in 1–1 correspondence with subrepresentations of $\ker\epsilon$. When $B = \mathbb{R}_q^{1,3}$ it is known from [21] that the double-bosonisation is the q-conformal group and the action on B is a q-deformation of its action on $\mathbb{R}^n$. Classically, this representation has one irreducible subrepresentation, the space of solutions of the massless Klein–Gordon equation. Further details will presented elsewhere [22].

Braided bicovariant calculi on the braided groups obtained by transmutation of quantum groups should also be looked at; being braided-commutative [23], they may have more natural exterior algebras. This would be analogous to the situation in Section 2, where we recalled that the braided version of the 'quantum Lie bracket' is better behaved for constructing some kind of enveloping algebra. This remains a direction for further work.

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